The Formal Definition of the Limit

Definition: Let \( f \) be a function defined on an open interval containing \( a \) (possibly undefined at \( a \) itself). Then, \( \lim_{x \to a} f(x) = L \) if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that, whenever \( 0 < |x - a| < \delta \), \( |f(x) - L| < \varepsilon \).

Okay, so what does that mean?
We know that a limit, \( L \), should be a value that \( f(x) \) is getting close to.
  i.e., \( |f(x) - L| \) should get small.
When? When \( x \) is close to \( a \).
  i.e., when \( |x - a| \) is small.

So, the definition says: First choose how close to \( L \) you want \( f(x) \) to be (i.e., within a distance \( \varepsilon \)). Then, if \( L \) really is the limit, you’ll be able to find how close \( x \) needs to be to \( a \) (i.e., a maximum acceptable distance \( \delta \)).

Note: Keeping \( |x - a| > 0 \) means you’re not allowed to use \( x = a \) itself (i.e., \( x \) is close to, but not equal to, \( a \)).

More explanation
Let’s look at a limit we’ve already seen. In Ex. 2 in Section 1.2, we saw that
\[
\lim_{x \to 1} \frac{1 - \sqrt{x}}{x - 1} = -0.5
\]

The formal definition says the function value \( f(x) \) can get as close to \( L = -0.5 \) as you want by letting the \( x \) value get sufficiently close to \( a = 1 \).

Let’s say you choose \( \varepsilon = 0.05 \). This means you want \( |f(x) - (-0.5)| < 0.05 \).
Algebraically, this is equivalent to \( -0.55 < f(x) < -0.45 \).

To visualize this, on a graphing calculator enter \( y_1 = (1 - \sqrt{x})/(x - 1) \), \( y_2 = -0.55 \), and \( y_3 = -0.45 \) with a window size of \([0, 2] \times [-1, 0]\).

Using an “intersect” function with \( y_1 \) and \( y_2 \) gives \( x \approx 0.6694 \) and with \( y_1 \) and \( y_3 \) gives \( x \approx 1.4938 \). These give \( |x - a| \) distances of
\[
|0.6694 - 1| = 0.3306 \quad \text{and} \quad |1.4938 - 1| = 0.4938
\]

These distances are shown in the figure at the top of the next page.
To satisfy both, I’ll choose $\delta = 0.33$.
Then $|x - a| < 0.33$ (i.e., $0.67 < x < 1.33$) guarantees that $|f(x) - L| < 0.05$

What if you chose $\varepsilon = 0.01$?
Then, $y_2 = -0.5 - 0.01 = -0.51$ and $y_3 = -0.5 + 0.01 = -0.49$. These gives intersection values of $x \approx 0.9231$ and $x \approx 1.0833$, respectively (which give distances of 0.0769 and 0.0833). To satisfy both, I’ll choose $\delta = 0.07$.
Then $|x - a| < 0.07$ (i.e., $0.93 < x < 1.07$) guarantees that $|f(x) - L| < 0.01$

The definition says that, if $-0.5$ really is the limit $L$, then no matter what $\varepsilon$ you pick, a $\delta$ can be found that works. From the graph for this example, you can see that no matter how small you make $\varepsilon$, you can always find a $\delta$ that will work.

Now, let’s look at a case where we can see the limit does not exist:

Let $g(x) = \frac{x^2 + x}{4|x|}$.

Sketch the graph on $[-8, 8]$ x $[-2, 2]$ and you’ll see a jump discontinuity at $x = 0$.

From the graph, $\lim_{x \to 0^-} g(x) = -0.25$ and $\lim_{x \to 0^+} g(x) = 0.25$.
\[ \lim_{x \to 0} g(x) \text{ D.N.E.} \]

But, suppose I had made a mistake and thought $L = 0.25$. We’ll try and follow the $\varepsilon$-$\delta$ approach using the given $g(x)$ and $a = 0$: 
You choose $\varepsilon = 1$. So, graph $y_1 = g(x)$, $y_2 = 0.25 - 1 = -0.75$, and $y_3 = 0.25 + 1 = 1.25$.

Note that $y_1$ and $y_2$ do not intersect, while $y_1$ and $y_3$ intersect at $x = -6$ and $x = 4$. So, I choose $\delta = 4$. Then $|x - a| < 4$ (i.e., $-4 < x < 4$) guarantees that $|g(x) - L| < 1$.

Does this one $\varepsilon$-$\delta$ pair satisfy the conditions of the definition?

NO... this needs to work for every $\varepsilon$.

Now, you choose $\varepsilon = 0.1$, giving $y_2 = 0.15$ and $y_3 = 0.35$. (Zoom in for a better graph.)

Note that $y_1$ and $y_2$ intersect at $x = -1.6$, and observe that for every $x$ in the interval $(-1.6, 0)$, the function is below $y = 0.15$.

This means that $|f(x) - L| \geq 0.1$.

So, no matter what I pick for $\delta$, the interval $|x - a| < \delta$ (i.e., $-\delta < x < \delta$) will contain values of $x$ where $|f(x) - L| \not< 0.1$. This proves that 0.25 is not the limit.

Given the separation in the graph at $x = 0$ (a vertical distance of 0.5), it seems reasonable that no matter what someone might guess for $L$, choosing $\varepsilon = 0.1$ will always lead to no possible value for $\delta$ (i.e., the limit does not exist).
Examples Using the Definition

Ex 1. Use the $\varepsilon$-$\delta$ definition to prove that $\lim_{x \to 3} 2x - 5 = 1$.

This means that we can’t just pick a few values for $\varepsilon$, but instead show that it works for every $\varepsilon > 0$.

We’ll start with some preliminary algebraic work:

$$
|f(x) - L| < \varepsilon \iff |(2x - 5) - 1| < \varepsilon \iff |2x - 6| < \varepsilon
$$

$$
\iff 2|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{2}
$$

Now we’re ready to prove the limit.

Proof: Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{2}$.

Then $0 < |x - a| < \delta$ $\Rightarrow$ $|x - 3| < \frac{\varepsilon}{2}$

$\Rightarrow$ $2|x - 3| < \varepsilon$

$\Rightarrow$ $|2x - 6| < \varepsilon$

$\Rightarrow$ $|(2x - 5) - 1| < \varepsilon$

$\Rightarrow$ $|f(x) - L| < \varepsilon$

(This proves the limit because it works for every $\varepsilon$.)

Ex 2. Use the $\varepsilon$-$\delta$ definition to prove that $\lim_{x \to 0} \frac{3}{\sqrt{x}} = 0$.

Preliminary algebraic work:

$$
|f(x) - L| < \varepsilon \iff \left| \frac{3}{\sqrt{x}} - 0 \right| < \varepsilon \iff \left| \frac{3}{\sqrt{x}} \right| < \varepsilon
$$

$$
\iff |x| < \varepsilon^3 \iff |x - 0| < \varepsilon^3
$$

Proof: Let $\varepsilon > 0$. Choose $\delta = \varepsilon^3$.

Then $0 < |x - a| < \delta$ $\Rightarrow$ $|x - 0| < \varepsilon^3$

$\Rightarrow$ $|x| < \varepsilon^3$

$\Rightarrow$ $\frac{3}{\sqrt{x}} < \varepsilon$

$\Rightarrow$ $\left| \frac{3}{\sqrt{x}} - 0 \right| < \varepsilon$

$\Rightarrow$ $|f(x) - L| < \varepsilon$
Ex 3. Use the $\varepsilon$-$\delta$ definition to prove that \( \lim_{x \to 3} 4x^2 - 49 = -13 \).

Preliminary algebraic work:
\[
|f(x) - L| < \varepsilon \iff |(4x^2 - 49) - (-13)| < \varepsilon \iff |4x^2 - 36| < \varepsilon \\
\iff 4|x^2 - 9| < \varepsilon \iff 4 \cdot |x + 3| \cdot |x - 3| < \varepsilon \\
\iff |x - 3| < \frac{\varepsilon}{4|x + 3|} \quad \text{(assuming } x \neq -3) 
\]

Note that the right-hand side depends on $x$! We need a constant.

So, since we’re interested in values of $x$ close to 3, let’s assume $2 < x < 4$

This means that $\delta \leq 1$. Then
\[
2 < x < 4 \implies 5 < x + 3 < 7 \implies 5 < |x + 3| < 7 \\
\implies 20 < 4|x + 3| < 28 \\
\implies \frac{1}{28} < \frac{1}{4|x + 3|} < \frac{1}{20} \\
\implies \frac{\varepsilon}{28} < \frac{\varepsilon}{4|x + 3|}
\]

Proof: Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{28}\right\}$.

Then $0 < |x - a| < \delta$ \implies $|x - 3| < \frac{\varepsilon}{28}$ and $2 < x < 4$
\[
\implies |x - 3| < \frac{\varepsilon}{4|x + 3|} \\
\implies 4|x + 3| |x - 3| < \varepsilon \\
\implies |4x^2 - 36| < \varepsilon \\
\implies |(4x^2 - 49) - (-13)| < \varepsilon \\
\implies |f(x) - L| < \varepsilon 
\]