Math 104 Section 2.3, The Train Problem

An engine pulls a train of mass 300 metric tons on a horizontal track at a constant power of 300 kilowatts. The resistances to the movement consists of the sum of two components: one is a constant and the other is proportional to the speed of the train. In an experiment, it is established that at a speed of 40 km/hr the resistance is 11,772 newtons, and at a speed of 24 km/hr, the resistance is 8829 newtons. Calculate the maximum speed of the train and the time to reach a speed of half the maximum speed. Assume the train starts from rest. Note that Power = Force × velocity.

Units

The given data require paying attention to the units. Note that the standard metric units are newtons, kilograms, meters, and seconds.

A metric ton is 1000 kg.
A newton is $\frac{\text{kilogram meter}}{\text{sec}^2}$.
A kilowatt is 1000 newton × $\frac{\text{meter}}{\text{sec}}$.

- **Power**

```
p = 300 * 1000
```

300 000

- **Mass**

```
m = 300 × 1000
```

300 000

- **Speed**

To convert the speed terms from km/hr to m/sec, multiply by

$$1000 \text{ m/sec} \times \frac{1 \text{ hour}}{3600 \text{ sec}}$$

```
v1 = 40 * 10/36
```

$\frac{100}{9}$

```
v2 = 24 * 10/36
```

$\frac{20}{3}$

Setting up the Differential Equation

The differential equation is simply $\sum$ Forces = mass × acceleration and there are two forces, the force due to the train engine and the force due to resisistance.

$$m \frac{dv}{dt} = \text{train engine} - \text{resistance} \tag{1}$$

The Train Engine

The power provided by the train engine is a constant **p** and we know force = $\frac{\text{power}}{\text{velocity}}$. Therefore,

$$\text{train engine} = \frac{p}{v} \tag{2}$$

The Resistance Force

The resistance force is known to consist of a constant and a term proportional to the velocity.

$$\text{resistance} = c + kv \tag{3}$$

Since we know a velocity of $v_1 = 40 \frac{\text{km}}{\text{hr}}$ has a resistance of 11,772 newtons, and a velocity of $v_2 = 24 \frac{\text{km}}{\text{hr}}$ has a resistance of 8829 newtons, we can set up two equations and two unknowns to solve for c and k

```
b1 = Solve[{11 772 == cc + kk * v1, 8829 == cc + kk * v2}, {cc, kk}]
```

$$\left\{\left\{cc \to \frac{8829}{2}, kk \to \frac{26\,487}{40}\right\}\right\}$$

In order to use these values, we next assign the values to the constants *c* and *k*.

```
c = b1[[1, 1, 2]]
```

$$\frac{8829}{2}$$

```
k = b1[[1, 2, 2]]
```

$$\frac{26\,487}{40}$$

Solving the Initial Value Problem

Putting the above information together, we have

$$\begin{aligned} \frac{m\, dv}{dt} &= \frac{p}{v} - (c + k\, v) \\ &= \frac{p - c\, v - k\, v^2}{v} \end{aligned} \tag{4}$$

Separating the variables gives

$$\int \frac{\text{dv}\, m\, v}{c\, v + k\, v^2 - p}\, dv = \int -1\, dt \qquad (5)$$

The expression inside the integral on the left side gives

$$\frac{\mathbf{m\, v}}{\mathbf{k\, v^2 + c\, v - p}}$$

$$\frac{300\,000\, v}{\frac{26\,487\, v^2}{40} + \frac{8829\, v}{2} - 300\,000}$$

$$\mathbf{b2} = \int \frac{m\, v}{c\, v + k\, v^2 - p}\, dv$$

$$4\,000\,000 \left(\frac{\left(40\,981 - 3\sqrt{4\,466\,929}\right) \log\left(-981\, v + 10\sqrt{4\,466\,929} - 3270\right)}{723\,642\,498} + \frac{\left(40\,981 + 3\sqrt{4\,466\,929}\right) \log\left(981\, v + 10\sqrt{4\,466\,929} + 3270\right)}{723\,642\,498} \right)$$

Since the initial condition is $v(0) = 0$, and since the right hand side of (5) is $-t$ + C, we get the constant of integration C by substituting $v = 0$ into **b2.**

```
myc = N[b2 /. v → 0]
```

4517.25

This gives the final solution as

```
finalans = Simplify[b2] == -t + myc
```

$$-\frac{1}{361\,821\,249} 2\,000\,000 \left(\left(3\sqrt{4\,466\,929} - 40\,981\right) \log\left(-981\, v + 10\sqrt{4\,466\,929} - 3270\right) - \left(40\,981 + 3\sqrt{4\,466\,929}\right) \log\left(981\, v + 10\sqrt{4\,466\,929} + 3270\right) \right) = 4517.25 - t$$

If you prefer a decimal version, use

```
finalansdec = N[finalans] // Simplify
```

$$191.478 \log(17\,865.1 - 981.\, v) + 1.\, t + 261.574 \log(981.\, v + 24\,405.1) = 4517.25$$

Finding to Maximum Velocity

To find the maximum velocity, one possible method is to solve for v and take the limit as $t \to \infty$. Attempting to solve for v leads to no results.

```
Solve[finalansdec, v]
```

Solve::tdep :
The equations appear to involve the variables to be solved for in an essentially non-algebraic way. ≫

Solve[191.478 log(17 865.1 − 981. v) + 1. t + 261.574 log(981. v + 24 405.1) = 4517.25, v]

Since this (and other commands) fails to produce any results, consider an alternative approach.

The train started from rest and has an increasing velocity. Therefore the maximum velocity should occur when the slope field has a horizontal asymptote. To find this, set the differential equation (4) equal to zero.

```
b4 = NSolve[p - c v - k v^2 == 0, v]
```

{{$v \to -24.8778$}, {$v \to 18.2111$}}

The positive value is in meter/second. Converting to km/hr gives

```
maxspeedftpersec = b4[[2, 1, 2]]
maxspeedkmperhr = maxspeedftpersec * 36/10
```

18.2111

65.56

Time to Reach Half of Maximum Speed

To determine the time it takes to reach half of the maximum speed, substitute the half of the maximum speed $\frac{\text{maxspeedftpersec}}{2}$ into **finalansdec** and solve for *t*. This time is given in seconds.

```
NSolve[finalansdec /. v → maxspeedftpersec/2, t]
```

{{$t \to 51.1389$}}